SOME ISSUES ON TOPOLOGICAL OPERATORS VIA IDEALS

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ABSTRACT. In a topological structure (X, τ) any ideal I of a subset of X induces a new topology τI (I). If Y is a regular structure, then under some assumptions on I, for any upper τ∗ quasi continuous multivalued map F : X → P(Y ) \ ∅, the sets of all points at which F is lower quasi-continuous (lower semi-continuous with respect to τ∗ or τ coincides). If F is compact-valued lower τ∗-quasi continuous, then the symmetrical result holds (J.Ewert [8]). The paper concerns operators in ideal topological structures. Some properties and characterizations of the set operator ( )∗ are investigated and explored. We define, explore, and analyze nano semi-local function as well as some of its characteristics. Finally, we design a graph of vital places in our city with regard to ideals.

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1. INTRODUCTION

Kuratowski has studied ideals in topological structures since 1930 [1]. The study written by Vaidyanathaswamy [2] in 1945 helped to establish the topic’s significance.

In the rich and nuanced landscape of topology, the concept of ideals plays a pivotal role in shaping and refining various topological structures. The interplay between ideals and topology gives rise to a multitude of new perspectives and tools, enhancing our ability to analyze and manipulate topological spaces. This paper, "Topological Operators Via Ideals," embarks on an in-depth exploration of how ideals influence and define topological operators, revealing the profound impacts and applications of these interactions [21-28].

An ideal I of a subset of a topological space (X, τ) introduces a modified topology τ(I), fundamentally altering the way we perceive continuity, convergence, and other topological properties. By leveraging these induced topologies, we can develop a richer understanding of multivalued maps, set operators, and function characteristics within these redefined spaces. The investigation of upper τ∗ - quasi continuous multivalued maps F : X → P(Y ) \ ∅ is a regular structure, stands at the forefront of this exploration. We aim to discern the conditions under which these maps exhibit lower quasi-continuity, specifically lower quasi-continuity concerning τ∗ or τ, thereby enhancing our comprehension of continuous behavior in ideal-induced topologies.

Building upon the foundational work of J. Ewert, this paper further explores the symmetrical properties of compact-valued lower τ∗-quasi continuous maps, offering a comprehensive view of continuity in the context of ideals. This dual examination of upper and lower quasi-continuity sheds light on the intricate balance maintained by these topological constructs.

In addition to the study of continuity, this paper delves into the properties and characterizations of the set operator F : X → P(Y ) \ ∅ within ideal topological structures. By thoroughly analyzing this operator, we uncover new dimensions of its behavior and implications, contributing to a deeper theoretical understanding of topological operations influenced by ideals.

The introduction of the nano semi-local function marks another significant advancement in this research. By defining and examining these functions, we reveal their unique characteristics and potential applications, further expanding the toolkit available.
for topological analysis.

To illustrate the practical relevance of these theoretical findings, we design a graph representing vital places in a city, informed by the underlying ideals. This practical application not only demonstrates the real-world utility of ideal topological structures but also bridges the gap between abstract theoretical concepts and tangible implementations.

In summary, "Topological Operators Via Ideals" offers a thorough and insightful investigation into the ways ideals transform topological operators. Through a blend of theoretical exploration and practical application, this paper enriches the field of topology with new properties, characterizations, and innovative approaches to understanding and utilizing topological spaces influenced by ideals.

Subsequently, numerous mathematicians employed the notion of ideals in topological structures. (see [3,4,10, 13,14,15,16,17]).

The use of ideals in general topology received a major boost by Newcomb in 1967 and by Berri, Porter and Stephenson in a surrey paper in 1968. The role of ideals in topological structures is a tool like filters. An innovative approaches to understanding and utilizing

implementations.

This closed set is written simply by $K^*_1(I)$ or even by $K_1^*$ if there is no possibility of confusion. $K^*_1 \subseteq Cl(K_1)$ is evident. It is quite easy to see that $K^*_1(\tau_2, I_2) \subseteq K^*_1(\tau_1, I_1)$ whenever $\tau_1 \subseteq \tau_2$ and $I_1 \subseteq I_2$.

The following fundamental characteristics of local function operator ($\tau^*$)are also clearly demonstrated:

1. $K_1^* \cup B^* = K_1^* \cup B^* = \emptyset, \quad (G \cap E) \cap K_1^* \subseteq ((G \cap E) \cap K_1)^*$

2. $K_1^* \subseteq K_1^*$ for any $K_1, B \subseteq X, G \subseteq \tau$ and $E \subseteq I$.

The family $\tau(I) = \{K_1 \subseteq X : K_1 \cap \tau(K_1)^* = \emptyset\}$ is a topology on $X_1$ better than $\tau$. A subset $K_1$ is closed in $(X_1, \tau^*(I))$ iff $K_1^* \subseteq K_1$. Consequently, the closure and interior operators in $(X_1, \tau^*(I))$ satisfy easily, $Cl^*(K_1) = K_1^* \cup K_1^*$ and Int$(K_1) = K_1 \cap (X_1 \setminus K_1)^*$.

The topological structure $\tau^*(I)$ may lack interest due to its discrete nature, as exemplified by the following example.

For example, if $X_1$ be the set of natural numbers, and $\tau$ be defined as follows: $\tau = \{\emptyset, [1,2], [1,2,3],...\}$.

The concept of an $\mathcal{H}$ was defined by Lellis Thivagar and Carmel Richard [3] which was defined using approximations and boundary region of a subset of an universe using an $ER$ on it and also they defined $n$ closed sets $n$ interior and $n$ closure. The concept of nI structures was introduced by Parimala et al. [4] and investigated its characteristics and properties. This study investigates and explores various properties and characterizations of the set operator ($\tau^*$). This work examines operators in nI/rs. We introduce the relationships between some weak forms of nopen sets in nI structures and some weak forms of nopen sets in nR structures. Also, we point out that the class of npl open sets is properly places between the classes of npl open and npl open sets. We give a decomposition of npl-continuity by proving that a mapping $f_i : (X_1, \tau, I) \rightarrow (Y_1, \sigma)$ is npl-continuous iff it is npl-continuous and npl-l-continuous.

Before entering in our work, we mention the next definitions.

2. PRELIMINARIES

Throughout this entire article, we will refer to $U_1$ and $V_1$ as an initial universal sets, $U/R$ and $V/R$ are equivalence classes (EC), $R$ and $R$ are equivalence relations (ER), $X_1 \subseteq U_1, Y_1 \subseteq V_1$.

DEFINITION 2.1.

A subset $A_i$ of a topological structure $(X_i, \tau)$ is called

1. Semi-open [7], if $A_i \subseteq Cl(\text{Int}(A_i))$.

2. Semi-closed [5,6], if $X_i \setminus A_i$ is semi-open.

The union of all semi-open sets contained in $A$ is denoted as $\text{slnt}(A)$ and is known as the semi-interior of $A$. The intersection of all semi-closed sets containing $A$ is denoted by $\text{sCl}(A)$ and is known as the semi-closure of $A$ [5].

DEFINITION 2.1. [6]

Consider Fig. 1

Assume that $(U_1, R)$ be an approximation structure and $X_1 \subseteq U_1$. Then:

The lower approximation $(L_{app}(X_1))$ and upper approximations $(H_{app}(X_1))$, since

$L_{app}(X_1) = \delta x \in U_1 | x_1 \in U : [x_1] R \subseteq X_1$, $H_{app}(X_1) = ...$
Sx ∈ U | x ∈ X : [x1]R n X1 ≠ φ.
and the boundary region Bn(X1) that is
BR(X1) = HR(X1) – Lapp(X1).

According to Pawlaks definition, X1 is called
a rough set if Hα(X1) = Lapp(X1).

The following two properties are equivalent for
any ideal topological structure
(X1, τ, I) {P1} K1 ∈ I ief K1(I) = ∅ ief K1 ∩ K1(I) = ∅. (P2) If [K1x : x ∈ V] is a family of sets belonging to I and
each K1x is an open set in the subspace \cup[K1x : x ∈ V], then
\cup[K1x : x ∈ V] ∈ I.

PROOF (NECESSITY)
Let (P1) be satisfied and let \{K1x : x ∈ V\} be a family of sets belonging to I since each K1x is open in \cup[K1x : x ∈ V]. For each point x ∈ \cup[K1x : x ∈ V] there exist x, ∈ V and open set G in (X1, τ) since x ∈ G n \cup[K1x : x ∈ V] = K1x ∈ I. Thus \cup[K1x : x ∈ V] n \cup[K1x : x ∈ V] = ∅, which implies \cup[K1x : x ∈ V] ∈ I.

PROOF (SUFFICIENCY)
Let (P2) hold. Since the implications
K1x ∈ I \Rightarrow K1x(I) = ∅ and

\frac{\text{Fig. 1. A rough set in a rough approximation structure.}}{\text{Fig. 1. A rough set in a rough approximation structure.}}
\( K^*_1(I) = \emptyset \Rightarrow K_1 \cap K_1^*(I) = \emptyset \) are true, it suffices to prove \( K_1 \cap K_1^*(I) = \emptyset \Rightarrow K_1 \subseteq I \). If \( K_1^\cap K_1^*(I) = \emptyset \), then each open set is semi-open. Therefore, there exists a point \( x \in \mathbb{Q} \) such that \( G \ni x \in \mathbb{K}_1 \). Then \( \{G \cap \mathbb{K}_1 : x \in \mathbb{K}_1\} \) is a family of sets belonging to \( I \) and each \( G \cap \mathbb{K}_1 \) is open in the subspace \( \mathbb{U}[G \cap \mathbb{K}_1 : x \in \mathbb{K}_1] \). Form the K-union we obtain \( K_1 = \bigcup[G \cap \mathbb{K}_1 : x \in \mathbb{K}_1] \in I \). Completed proof.

**Remark 2.1.**

If an ideal \( I \) satisfies \((P_1)\), then \( \beta(I) = \tau(I) \) (where \( \beta(I) = \{G \in \mathbb{E} : G \in \tau, E \in I\} \) is a base of the topology \( \tau(I) \) in \( X_i \)). [8], this equality is not necessary, as shown by the following example due to Ewert [8].

**Example 2.1.**

If \((R, U)\) is usual topology and \( I \) is the ideal of all bounded subsets of \( \mathbb{Q} \subseteq \mathbb{R} \), then \( R \cap Q = \cup\{(-n, n) : \mathbb{Q} = 1, 2, ..., \infty\} \in \tau(I) \) and \( R \cap \beta(I) \).

**Definition 3.1.**

The semi-local function [14] of an ideal \( I \) with respect to \( \tau \) is defined in \( P(X) \) as:

\[ K^*(I) = \{x \in X : G \ni x \not\in I \text{ for each } G \in \sigma(X, \tau)\} \]

writing simply \( K^*_1(I) \) for \( K^*(I, \tau) \).

**Remark 3.1.**

If an ideal \( I \) has the property \((P_1)\), then \( K_1^1 \cap K_1^*(I) = \text{S} \mathbb{C} \mathbb{L}(K_1) \) for each set \( A \subseteq X_i \).

**Proposition 3.1.**

Let \((X_i, \tau, I)\) be an ideal topological structure and \( K_i \subseteq X_i \). Then:

1. \( K^*_1(I) \subseteq K^*_1(I, \tau) \subseteq X_i \)
2. \( K^*_1(I) = K^*_1(I, \tau) \) if \( \tau = \sigma(X_i, \tau) \).
3. If \( K_i \subseteq I \), then \( K^*_1(I) = \emptyset \).
4. In general, neither \( K^*_1(I) \cap K^*_1(I) \subseteq K^*_1(I) \).

**Proof.**

The statements (2) - (4) are a direct result of the definitions, now, let \( x \in K^*_1(I) \). Then from Definition 3.1, \( K_i \cap G \in I \) for each \( G \in \sigma(X_i, \tau) \). Since each open set is semi-open. Therefore, \( x \in K^*_1(I) \) and (1) is proved.

**Example 3.1.**

The converse of Statement (1) from Proposition 3.1 is not generally true. Let \( X_i = \{1, 2, 3, 4\} \) and \( \tau = \{X, \phi, [1, 2]\} \) with \( J = \{\phi, [1, 2], [1, 2]\} \). Set \( K_i = \{1, 2\} \) and then \( K^*_1(I) = [3, 4] = \mathbb{C} \mathbb{L}(K^*_1(I)) \) and \( K^*_1(I) = \{4\} = \mathbb{C} \mathbb{L}(K^*_1(I)) \).

**Lemma 3.1.**

If an ideal \( I \) in a topological structure \((X_i, \tau)\) satisfies property \((P_2)\), then \( \forall K_i \subseteq X_i \), the sets \( \mathbb{C} \mathbb{L}(K_i) \setminus \mathbb{S} \mathbb{C} \mathbb{L}(K_i) \) and \( \mathbb{S} \mathbb{T}(K_i) \setminus \mathbb{T}(K_i) \) are nowhere dense in \( (X_i, \tau) \).

**Proof.**

Let \( x \in \mathbb{R} \) and let \( x \not\in I \) is a semi-open. Therefore, there exists \( G \subset \mathbb{Q} \setminus \emptyset \) for each \( G \in \sigma(X, \tau) \). Therefore, there exists a point \( y \in (K_i)^\cap(I) \cap \mathbb{Q} \). Since \( y \in K^*_1(I) \), \( K_i \cap G \not\in I \) and hence \( x \in K^*_1(I) \).

Thus, we have \( \mathbb{S} \mathbb{C} \mathbb{L}(K^*_1(I)) \subseteq K^*_1(I) \).

Also, let \( x \in \mathbb{S} \mathbb{C} \mathbb{L}(K^*_1(I)) = K^*_1(I) \), then \( G \cap

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Let $X, \tau$ be a topological structure with two $K$ ideals $I, J$ on $X$ and $A$ be a subset of $X$. Then, the next properties hold:

1. If $I \subseteq J$, then $K_1^{ss}(I) \subseteq K_1^{ss}(J)$;
2. $K_1^{ss}(I \cap J) = K_1^{ss}(I) \cap K_1^{ss}(J)$.

**Proof.**

Let $I \subseteq J$ and $x \in K_1^{ss}(I)$. Then $x \in K_1^{ss}(J)$.

4. Some Characterizations of Nano Semi-local Functions

The nano local function of a nlf with regard to when it is defined on $P(X)$ as:

4.2. If $I$ satisfies (P1), then $\beta(I) = \tau(I)$ (where $\beta(I) = \{G \in I : G \subset I \}$ or equivalently, $\beta(I) = \{G \in I : G \subset I \}$).

**Proposition 4.1.**

The following two properties are equivalents for any nlf structure $(X, \tau, I)$:

$(P_1)$ $I \subseteq I$ is equivalent to $(P_2)$ $I \subseteq I$.

**Remark 2.1.**

If an nlf satisfies (P1), then $\beta(I) = \tau(I)$ (where $\beta(I) = \{G \in I : G \subseteq I \}$ is a base of the nlf $\tau(I)$ in $X$, [8], this equality is not necessary, as shown by the following example due to Ewert [8].

**Definition 4.2.**

The nano semi-local function [14] of a nlf $I$ with respect to $\tau$ is defined on $P(X)$ as: -
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LEMMA 4.3.

If a $nl$ has the property $(P_i)$, then $K_i \cup K_i^{**} = sCl(K_i)$ for each set $A \subseteq X_1$.

PROPOSITION 4.4.

Let $(X_i, \tau, I)$ be an $nl$ and $K_i \subseteq X_1$ thus.

(1) $K_i^{**} \subseteq K_i^*$, if $\forall K_i \subseteq X_1$.

(2) $K_i^{**}(\tau) = K_i^*$ if $\tau = sO(X_i, \tau)$.

(3) For each $K_i \subseteq X_1$, then $K_i = \emptyset$.

(4) In general, neither $\beta \subseteq K_i^{**}(I)$ nor $K_i^{**}(I) \subseteq K_i$.

REMARK 4.5.

If a $nl$ in a $n$-structure $(X_i, \tau)$ satisfies property $(P_i)$, then $\forall K_i \subseteq X_1$, the sets $Cl(K_i)\backslash sCl(K_i)$ and $sInt(K_i)\backslash Int(K_i)$ are nowhere dense in $(X_i, \tau)$.

PROPOSITION 4.6.

Let $(X_i, \tau, I)$ be a $nl$ structure and $K_i, B \subseteq X_1$. The following characteristics hold for the nano semi-local function:

(a) $K_i^{**}(I) = sCl(K_i^{**}(I)) \subseteq sCl(K_i^*)$.

(b) $K_i^{**}(I) \subseteq K_i^*$.

(c) If $K_i \subseteq B$, then $K_i^{**}(I) \subseteq B^{**}(I)$.

(d) $(K_i \cup B)^{**}(I) = K_i^{**}(I) \cup B^{**}(I)$.

(e) $(K_i^{**})^{**}(I) \subseteq K_i^{**}(I)$.

(f) $K_i^{**}(I)$ is a semi-closed set in $X_1$.

(g) If $E \subseteq I$, then $(K_i \backslash E)^{**}(I) \subseteq K_i^{**}(I)$.

(h) $(K_i \backslash E)^{**}(I) = K_i^{**}(I)$ if $\tau = sO(X_i, \tau)$.

(i) If $G \subseteq \tau$, then $G \cap K_i^{**}(I) = G \cap (G \cap K_i^{**}(I)) \subseteq (G \cap K_i^{**}(I))$.

(j) If $I = \emptyset$, then $K_i^{**}(I) = sCl(K_i^*)$.

(k) If $I = P(X_i)$, then $K_i^{**}(I) = \emptyset$, for each $K_i \subseteq X_1$.

THEOREM 4.7.

Let $(X_i, \tau, I)$ be a topological structure with two $nl$ $I, J$ on $X_i$ and $A$ be a subset of $X_i$. Then, the next properties hold:

(1) If $I \subseteq J$, then $K_i^{**}(J) \subseteq K_i^{**}(I)$.

(2) $K_i^{**}(I \cup J) = K_i^{**}(I) \cup K_i^{**}(J)$.

5. CONCLUSIONS

The development of nano ideal topology in mathematical structures of nano approximations has attracted many experts in many fields. In this work we have investigated the concept of semi-local functions due to the first author in [14]. Also, we observed that semi local functions are not a Kuratowski closure operator. We established some properties and characterizations of nano semi local functions. We introduced a real-life application based on nano semi local mappings. We established the best path for accessing the vital places in our city. Through the future work, we will investigate more applications for $nl$ topology such as human blood circulation and provide a topological technique for getting the vigour of different biological solutions.

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